MTH 132 Exam 2 Topics

June 2, 2014

Derivatives and Tangent Lines

You should know the definition of the derivative, and how it's related to the slopes of secant lines and tangent lines. If f is a function on the real line, then its derivative can be defined through either of

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z\to x} \frac{f(z) - f(z)}{z - x}$$

Remember that this is just a formalization of

$$\text{slope of } f = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

You should be able to compute the derivative of a function directly from the definition, for simple functions (e.g. polynomials, power functions, rational functions like $1/x$, and so on). As an example, we can compute the derivative of $\sqrt{x}$ using the second of the above formulas, since

$$\frac{\sqrt{z} - \sqrt{x}}{z - x} = \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} = \frac{1}{\sqrt{z} + \sqrt{x}}$$

for $z \neq x$. Taking a limit as $z \to x$, we find that the derivative is then $1/(2\sqrt{x})$.

Derivative Rules

You should know how to use the formulas we've written down for derivatives, such as

$$\begin{aligned}
(f+g)' &= f' + g' \\
(cf)' &= cf' \\
(fg)' &= fg' + f'g \\
\left(\frac{f}{g}\right)' &= \frac{gf' - fg'}{g^2} \\
(f(g))' &= f'(g)g'
\end{aligned}$$

You should also know how to differentiate some specific functions, such as

$$\frac{d}{dx} x^n = nx^{n-1}$$

and that

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

You might be asked to combine these rules in order to compute the derivative of a more complicated function. For example, suppose we want to differentiate

$$f(x) = \frac{\sin x^2}{x}$$

We can use the quotient rule to find that

$$\frac{df}{dx} = \frac{x\left(\frac{d}{dx} \sin x^2\right) - (\sin x^2) \cdot 1}{x^2}$$

In turn, the chain rule implies that

$$\frac{d}{dx}\sin x^2 = \cos x^2 \cdot \left(\frac{d}{dx}x^2\right) = \cos x^2 \cdot 2x$$

Putting it all together gives the desired derivative. Make sure to parse your work: Separate steps, and clearly identify which rules you're using. It'll help you avoid mistakes, and help you figure out exactly what rule to use.

Applications to Physics and other fields

Derivatives are useful exactly because they represent rates of change - whenever you see 'rate of change,' or 'how quickly — does —,' or similar phrases, you should start thinking about derivatives. As an example, we have the fundamental relationships

$$\text{velocity} = \frac{d}{dt}\text{ position}$$

$$\text{acceleration} = \frac{d}{dt}\text{ velocity}$$

Furthermore, we have speed = |velocity|. For example, if we have a spring with a weight attached, the motion of the weight might be modeled by something like

$$s(t) = 5\cos t$$

(this represents that the weight starts off 5 units from equilibrium, at its maximum displacement). If we want to know when the acceleration is zero, we compute

$$a(t) = \frac{d}{dt}\left(\frac{d}{dt}5\cos t\right) = \frac{d}{dt}(-5\sin t) = -5\cos t$$

Setting this to 0, we find that the acceleration is 0 precisely when the position is also zero - namely, at $t = \pi/2, 3\pi/2, 5\pi/2$, and so on. So the block has zero acceleration only when it's passing through equilibrium. Note that we can also make sense of units here:

$$\frac{ds}{dt} = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}$$

The quantity Δt has units of time, and Δs has units of distance - so the quotient (and hence the limit) has units distance / time. This matches the units of velocity.

We can make similar studies in other areas - whenever you have a rate of change, it's likely that it's related to a derivative. So in economics, the marginal cost (which can be thought of as describing the change in cost per additional unit) is the derivative of the cost function with respect to quantity.

The Chain Rule

The chain rule is the most important of all the derivative formulas we've discussed, but it's probably also the trickiest to apply. Remember that it says

$$\frac{d}{dx}f(g(x)) = \underbrace{f'(g(x))}_{\substack{\text{derivative of outside}\\ \text{evaluated at inside}}} \cdot \underbrace{g'(x)}_{\text{derivative of inside}}$$

This has a rather nice representation in terms of the Leibniz notation: If y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Do be careful: These **are not** fractions, and can't be treated as such.

We had two main uses so far for the chain rule: Implicit differentiation, and applying it for related rates problems.

Suppose we have a function where y is given implicitly in terms of x - perhaps something like

$$\sin(y^3 + y) = \cos x$$

Solving for y in this equation would be quite difficult, so we can't really write y as an explicit function of x. But we can differentiate both sides, using the chain rule:

$$\begin{aligned}
\frac{d}{dx}\sin(y^3 + y) &= \frac{d}{dx}\cos x \\
\cos(y^3 + y)\frac{d}{dx}(y^3 + y) &= -\sin x \\
\cos(y^3 + y)\left(3y^2\frac{dy}{dx} + \frac{dy}{dx}\right) &= -\sin x \\
\frac{dy}{dx} &= \frac{-\sin x}{\cos(y^3 + y)(3y^2 + 1)}
\end{aligned}$$

In general, there's a two step process:

- Differentiate an equation on both sides, noticing where dy/dx shows up.
- Solve the resulting (linear!) equation for dy/dx to find the derivative.

We also have *related rates* problems, where we are given information about how one (or more) variable is changing, and want to figure out how a related quantity behaves. For example, suppose we have a sphere with radius r, and r is decreasing at a rate of 2cm / s when the radius is $r = 12$cm. This gives us

$$\frac{dr}{dt} = -2$$

If we want to determine how quickly the surface areas is changing at this moment, we can use

$$A = 4\pi r^2 \implies \frac{dA}{dt} = 8\pi r\frac{dr}{dt}$$

Evaluating with the numbers above, we see that A is decreasing at a rate of 192cm^2 / s.

In general, the outline for a related rates problem is this:

- Draw a picture of the physical situation, and use it to introduce variables.
- Write down the information you know, converting all rates of change into statements about derivatives.
- Write down an equation relating the quantities you're interested in, and differentiate the equation with the chain rule as needed.
- Plug in numbers, and solve for the desired quantities.

There are several examples of physical problems in the textbook, §3.8.

Linearization

One of the nicest things about the derivative is that it gives the slope of the tangent line, which is a reasonable approximation to f near the point we're studying. Using this linearization, we can use easily-computed values of f to approximate harder ones. For example, if we want to compute $\sqrt{101}$, we can use the fact that $\sqrt{100} = 10$, together with the fact that

$$\left.\frac{d}{dx}\right|_{x=100}\sqrt{x} = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

So our approximation (again, using that $f'(x)\Delta x \approx \Delta f$) is that

$$\sqrt{101} = \sqrt{100} + \frac{1}{20}\cdot(101 - 100) = 10.05.$$

Plugging this into a calculator, $\sqrt{100} \approx 10.0499$, so this is a pretty good approximation.

In general, the linearization of f at a can be written as

$$L(x) = f'(a)(x-a) + f(a)$$

One way to remember this might be to write it as

$$f(x) - f(a) \approx f'(a)(x-a)$$

since this is $\Delta f \approx f'(a)\Delta x$.

This isn't a complete list of topics, or what can be covered on the exam, but it's a good place to start. Make sure that you can draw the pictures for these concepts - derivatives and related ideas have very important geometric interpretations which can help guide you. There are many problems in the textbook (from the Chapter 3 review), as well as WeBWorK that you can use to review; any material from sections $3.1 - 3.9$ is fair for the exam.